ON THE ALGEBRAIC INTEGRALS IN THE PROBLEM OF MOTION OF A RIGID BODY IN A NEWTONIAN FIELD OF FORCE

(OB ALGEBRAICHESKIKH INTEGRALAKH V ZADACHE O DVIZHENII TVERDOGO TELA V N'IUTONOVSKOM POLE SIL)

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It has been shown that in the problem of motion of a rigid body about a fixed point under the action of a Newtonian field of force the fourth algebraic integral exists only in the case analogous to Euler's case in the problem of motion of a rigid body about a fixed point in a homogeneous gravity field and in the case analogous to that of Lagrange.

1. As is well known [1], the approximate equations of motion of a rigid body about a fixed point under the action of a central Newtonian field of force

$$\begin{aligned}
&A\frac{dp}{dt}+(C-B)\,qr=y_0'\gamma''-z_0'\gamma'+\alpha(C-B)\gamma'\gamma'', &&\frac{d\gamma}{dt}=r\gamma'-q\gamma''\\
&B\frac{dq}{dt}+(A-C)\,pr=z_0'\gamma-x_0'\gamma''+\alpha(A-C)\gamma''\gamma, &&\frac{d\gamma'}{dt}=p\gamma''-r\gamma\\
&C\frac{dr}{dt}+(B-A)\,pq=x_0'\gamma'-y_0'\gamma+\alpha(B-A)\gamma\gamma', &&\frac{d\gamma''}{dt}=q\gamma-p\gamma'
\end{aligned}\qquad(1.1)$$

$$\left(x_0'=Mgx_0,\ y_0'=Mgy_0,\ z_0'=Mgz_0,\ \alpha=\frac{3g}{R}\right)$$

have three first independent algebraic integrals: the kinetic energy integral, the integral of areas, and the trivial integral

$$Ap^2+Bq^2+Cr^2-2(x_0'\gamma+y_0'\gamma'+z_0'\gamma'')+\alpha(A\gamma^2+B\gamma'^2+C\gamma''^2)=\text{const}\qquad(1.2)$$

$$Ap\gamma+Bq\gamma'+Cr\gamma''=\text{const},\qquad \gamma^2+\gamma'^2+\gamma''^2=1$$

The system of equations (1.1) does not contain the time t explicitly, besides it has the last Jacobi's multiplier equal to unity; therefore, the existence of the fourth integral, which has turned out to be algebraic, for the system (1.1)

$$A^2p^2 + B^2q^2 + C^2r^2 - \alpha(BC\gamma^2 + AC\gamma'^2 + AB\gamma''^2) = \text{const} \quad \text{for } x_0' = y_0' = z_0' = 0$$
$$r = \text{const} \quad \text{for } A = B, \ x_0' = y_0' = 0$$

in these two cases allows the problem to be reduced to quadratures. The question arises, in which other cases is the existence of the fourth algebraic integral possible. It is clear, that the condition proved in [2], that the ellipsoid of inertia has to be an ellipsoid of rotation, is the necessary but not the sufficient condition for the existence of the new fourth algebraic integral.

Let us show that for $A = B$ and $\alpha \neq 0$ the fourth algebraic integral is possible only if $x_0' = y_0' = 0$, which is an analogue of Lagrange's case in the classical problem of motion of a heavy rigid body about a fixed point.

2. Let us introduce new variables y_1, y_2, z_1, z_2 in place of p, q, γ, γ'

$$y_1 = p + iq, \qquad y_2 = p - iq, \qquad z_1 = \gamma + i\gamma', \qquad z_2 = \gamma - i\gamma' \qquad (i = \sqrt{-1})$$

and, replacing t by $-it$, let us rewrite the system (1.1) (assuming, of course, that $A = B$ and $y_0' = 0$) in the form

$$\begin{aligned} A\frac{dy_1}{dt} &= -(A - C)ry_1 + z_0'z_1 - x_0'\gamma'' + \alpha(A - C)\gamma''z_1 \\ A\frac{dy_2}{dt} &= (A - C)ry_2 - z_0'z_2 + x_0'\gamma'' - \alpha(A - C)\gamma''z_2 \\ 2C\frac{dr}{dt} &= x_0'(z_2 - z_1), \qquad \frac{dz_1}{dt} = y_1\gamma'' - rz_1, \\ \frac{dz_2}{dt} &= rz_2 - y_2\gamma'', \qquad 2\frac{d\gamma''}{dt} = y_2z_1 - y_1z_2 \end{aligned} \tag{2.1}$$

This system has the following first integrals

$$\begin{aligned} &Ay_1y_2 + Cr^2 - x_0'(z_1 + z_2) - 2z_0'\gamma'' - \alpha(A - C)\gamma''^2 = \text{const} \\ &A(y_1z_2 + z_1y_2) + 2Cr\gamma'' = \text{const}, \qquad z_1z_2 + \gamma''^2 = \text{const}. \end{aligned} \tag{2.2}$$

Now we introduce an arbitrary parameter λ by means of the substitution of $\lambda^2 y_1$, λy_2, λr, $\lambda^3 z_1$, $\lambda^2 z_2$, $\lambda^3 \gamma''$, t/λ in place of y_1, y_2, r, z_1, z_2, γ'', t; then the system (2.1) and its first integrals (2.2) are rewritten in the form

$$\begin{aligned} A\frac{dy_1}{dt} &= -(A - C)ry_1 + z_0'z_1 - x_0'\gamma'' + \lambda^3\alpha(A - C)\gamma''z_1 \\ A\frac{dy_2}{dt} &= (A - C)ry_2 - z_0'z_2 + \lambda x_0'\gamma'' - \lambda^3\alpha(A - C)\gamma''z_2 \end{aligned} \tag{2.3}$$

$$2C\frac{dr}{dt}=x_0'(z_2-\lambda z_1),\quad \frac{dz_1}{dt}=-rz_1+\lambda y_1\gamma'',\quad \frac{dz_2}{dt}=rz_2-\lambda y_2\gamma'',\quad 2\frac{d\gamma''}{dt}=y_2z_1-y_1z_2$$

$$\begin{gathered} Cr^2-x_0'z_2+\lambda(Ay_1y_2-x_0'z_1-2z_0'\gamma'')-\lambda^4\alpha(A-C)\gamma''^2=h_1 \\ A(y_1z_2+z_1y_2)+2Cr\gamma''=h_2,\qquad z_1z_2+\lambda\gamma''^2=h_3 \end{gathered} \tag{2.4}$$

where h_1, h_2, h_3 are certain arbitrary constants.

Considering the system of equations, obtained from the system (2.3) with $\alpha = 0$, Husson [3] has proved, that in the problem of motion of a solid about a fixed point in a homogeneous gravity field, in the case when the ellipsoid of inertia with respect to the fixed point is an ellipsoid of rotation, the fourth algebraic integral may exist only in the cases of Lagrange ($A = B$, $x_0' = y_0' = 0$) and Kowalewski ($A = B = 2C$, $y_0' = z_0' = 0$). The proof was carried out by using the first three terms of the expansion of the general integral of the obtained system of equations into a power series with respect to the parameter λ which was assumed to be small. Here, it was taken into consideration that the right-hand sides of the obtained system of equations and its first integrals are polynomials with respect to y_1, y_2, r, z_1, z_2, γ'' and λ.

Since the first three terms of the expansion of the general integral of the system (2.3) into a power series with respect to a small parameter λ do not depend on α, and the right-hand sides of the system (2.3) as well as the relations (2.4) also are polynomials with respect to y_1, y_2, r, z_1, z_2, γ'', λ, the Husson's result must be looked upon as the necessary condition for the existence of the new fourth integral of the problem under consideration.

3. Now let us find the necessary and sufficient conditions for the existence of the new fourth algebraic integral by showing that the system of equations (2.1), rewritten for the case $A = B = 2C$, $y_0' = z_0' = 0$ in the form

$$\begin{gathered} 2\frac{dy_1}{dt}=-ry_1-c\gamma''+\alpha\gamma''z_1,\qquad 2\frac{dy_2}{dt}=ry_2+c\gamma''-\alpha\gamma''z_2,\qquad 2\frac{dr}{dt}=c(z_2-z_1) \\ \frac{dz_1}{dt}=y_1\gamma''-rz_1,\qquad \frac{dz_2}{dt}=rz_2-y_2\gamma'',\qquad 2\frac{d\gamma''}{dt}=y_2z_1-y_1z_2\qquad \left(c=\frac{x_0'}{C}\right) \end{gathered} \tag{3.1}$$

has only three first algebraic integrals

$$\begin{gathered} 2y_1y_2+r^2-c(z_1+z_2)-\alpha\gamma''^2=\text{const} \\ y_1z_2+z_1y_2+r\gamma''=\text{const},\qquad z_1z_2+\gamma''^2=\text{const} \end{gathered} \tag{3.2}$$

Replacing in the system (3.1) the quantities

$$y_1,\ y_2,\ r,\ z_2,\ t \quad \text{by}\ \lambda^{1/2}y_1, \qquad \lambda^{-1/2}y_2, \qquad \lambda^{-1/2}r, \qquad \lambda^{-1}z_2, \qquad \lambda^{1/2}t$$

where λ is an arbitrary parameter, we obtain the system of equations

$$\begin{gathered} 2\frac{dy_1}{dt} = -ry_1 - c\gamma'' + \alpha\gamma'' z_1, \qquad 2\frac{dy_2}{dt} = ry_2 - \alpha\gamma'' z_2 + \lambda c\gamma'' \\ 2\frac{dr}{dt} = c(z_2 - \lambda z_1), \qquad \frac{dz_1}{dt} = -rz_1 + \lambda y_1\gamma'' \\ \frac{dz_2}{dt} = rz_2 - \lambda y_2\gamma'', \qquad 2\frac{d\gamma''}{dt} = y_2 z_1 - y_1 z_2 \end{gathered} \tag{3.3}$$

which is satisfied by the first algebraic integrals

$$\begin{gathered} r^2 - cz_2 + \lambda(2y_1y_2 - cz_1 - \alpha\gamma''^2) = h_1 \\ y_1z_2 + z_1y_2 + r\gamma'' = h_2, \qquad z_1z_2 + \lambda\gamma''^2 = h_3 \end{gathered} \tag{3.4}$$

Here h_1, h_2, h_3 are certain arbitrary constants, which are such functions of λ, that for $\lambda = 1$ the integrals (3.4) become the integrals (1.2) rewritten for this case. It follows from Husson's paper [3] that if the system (3.1) has the fourth algebraic integral, then the system

$$\frac{dy_2}{dr} = \frac{ry_2 - \alpha\gamma'' z_2 + \lambda c\gamma''}{c(z_2 - \lambda z_1)}, \qquad \frac{d\gamma''}{dr} = \frac{y_2 z_1 - y_1 z_2}{c(z_2 - \lambda z_1)} \tag{3.5}$$

in which the quantities y_1, z_1, z_2 have been replaced from (3.4), has an algebraic integral

$$F(h_1, h_2, h_3, y_2, \gamma'', r) = \text{const}$$

This integral can be expanded into a power series with respect to powers of $\lambda^{1/p}$ (p is an integer) in the neighborhood of the point $\lambda = 0$

$$F_0(y_2, \gamma'', r) + \lambda^{1/p} F_1(y_2, \gamma'', r) + \ldots + \lambda F_p(y_2, \gamma'', r) + \ldots = \text{const} \tag{3.6}$$

with coefficients which are algebraic functions of all of their arguments.

In this expansion F_0 necessarily depends upon at least one of the quantities y_2 or γ''. Besides, for λ sufficiently small, the general solution $y_2(r)$ and $\gamma''(r)$ of the system (3.5) can be expanded into series [4] of integral powers of the parameter λ

$$\gamma'' = \gamma_3^{(0)} + \lambda\gamma_3^{(1)} + \ldots, \quad y_2 = y_2^{(0)} + \lambda y_2^{(1)} + \ldots \tag{3.7}$$

Here the coefficients $\gamma_3^{(0)}$, $y_2^{(0)}$, $\gamma_3^{(1)}$, $y_2^{(1)}$ will be determined from the equations

$$\frac{dy_2^{(0)}}{dr} = \frac{1}{cz_2^{(0)}}\{ry_2^{(0)} - \alpha\gamma_3^{(0)}z_2^{(0)}\}, \qquad \frac{d\gamma_3^{(0)}}{dr} = \frac{1}{cz_2^{(0)}}\{y_2^{(0)}z_1^{(0)} - y_1^{(0)}z_2^{(0)}\} \tag{3.8}$$

$$\begin{aligned}\frac{dy_2^{(1)}}{dr} &= \frac{1}{cz_2^{(0)}}\Big\{ry_2^{(1)} - \alpha(\gamma_3^{(0)}z_2^{(1)} + z_2^{(0)}\gamma_3^{(1)}) + c\gamma_3^{(0)} - \\ &\qquad - \frac{1}{z_2^{(0)}}(z_2^{(1)} - z_1^{(1)})(ry_2^{(0)} - \alpha\gamma_3^{(0)}z_3^{(0)})\Big\} \\ \frac{d\gamma_3^{(1)}}{dr} &= \frac{1}{cz_2^{(0)}}\Big\{y_2^{(0)}z_1^{(1)} + z_1^{(0)}y_2^{(1)} - y_1^{(0)}z_2^{(1)} - z_2^{(0)}y_1^{(1)} - \\ &\qquad - \frac{1}{z_2^{(0)}}(z_2^{(1)} - z_1^{(0)})(y_2^{(0)}z_1^{(0)} - y_1^{(0)}z_2^{(0)})\Big\}\end{aligned} \tag{3.9}$$

The expansions of the functions

$$y_1 = y_1^{(0)} + \lambda y_1^{(1)} + \ldots, \quad z_1 = z_1^{(0)} + \lambda z_1^{(1)} + \ldots, \quad z_2 = z_2^{(0)} + \lambda z_2^{(1)} + \ldots \tag{3.10}$$

together with the expansion (3.7) must satisfy [5] the system (3.4).

The substitution of the expansion (3.7) into the formula (3.6) gives

$$\begin{aligned}F_0(y_2^{(0)}, \gamma_3^{(0)}, r) + \lambda^{1/p}F_1(y_2^{(0)}, \gamma_3^{(0)}, r) + \ldots + \lambda\Big[y_2^{(1)}\frac{\partial F_0(y_2^{(0)}, \gamma_3^{(0)}, r)}{\partial y_2^{(0)}} + \\ + \gamma_3^{(1)}\frac{\partial F_0(y_2^{(0)}, \gamma_3^{(0)}, r)}{\partial \gamma_3^{(0)}} + F_p(y_2^{(0)}, \gamma_3^{(0)}, r)\Big] + \ldots + \lambda^2(\ldots) + \ldots = \text{const}\end{aligned}$$

Due to this relation, the first integrals of the systems (3.8) and (3.9), which must exist in this case, can be represented in the following form, respectively

$$F_0(y_2^{(0)}, \gamma_3^{(0)}, r) = \text{const} \tag{3.11}$$

$$y_2^{(1)}\frac{\partial F_0(y_2^{(0)}, \gamma_3^{(0)}, r)}{\partial y_2^{(0)}} + \gamma_3^{(1)}\frac{\partial F_0(y_2^{(0)}, \gamma_3^{(0)}, r)}{\partial \gamma_3^{(0)}} + F_p(y_2^{(0)}, \gamma_3^{(0)}, r) = \text{const} \tag{3.12}$$

where the left-hand side of the expression (3.11) will necessarily depend upon at least one of the variables $y_2^{(0)}$ or $\gamma_3^{(0)}$.

4. To prove the statement made in Section 3 it is sufficient to demonstrate that the system (3.3) has a solution for which it is impossible to find the fourth algebraic integral. Let us consider the particular solution defined by the following values of the arbitrary constants in the formulas (3.4)

$$h_1 = a^2, \quad h_2 = \lambda^2, \quad h_3 = \lambda^2 \qquad (a \text{ does not depend on } \lambda)$$

Substituting the expansions (3.7) and (3.10) into the formulas (3.4) which for the particular solution under consideration are rewritten in the form

$$r^2 - cz_2 + \lambda(2y_1y_2 - cz_1 - \alpha\gamma''^2) = a^2, \quad y_1z_2 + z_1y_2 + r\gamma'' = \lambda^2, \quad z_1z_2 + \lambda\gamma''^2 = \lambda^3 \tag{4.1}$$

we will have the expressions

$$y_1^{(0)} = -\frac{cr\gamma_3^{(0)}}{r^2 - a^2}, \qquad z_1^{(0)} = 0, \qquad cz_2^{(0)} = r^2 - a^2 \tag{4.2}$$

and the relations for obtaining the quantities $y_1^{(1)}$, $z_1^{(1)}$, $z_2^{(1)}$

$$\begin{gathered} y_1^{(1)}z_2^{(0)} = -(y_1^{(0)}z_2^{(1)} + y_2^{(0)}z_1^{(1)} + r\gamma_3^{(1)}), \qquad z_1^{(1)}z_2^{(0)} = -(\gamma_3^{(0)2} - z_1^{(0)}z_2^{(1)}) \\ cz_2^{(1)} = 2y_1^{(0)}y_2^{(0)} - \alpha\gamma_3^{(0)2} - cz_1^{(0)} \end{gathered} \tag{4.3}$$

Then the equations (3.8), which by means of (4.2) are represented in the form

$$\frac{dy_2^{(0)}}{dr} = \frac{r}{r^2 - a^2}y_2^{(0)} - \frac{\alpha}{c}\gamma_3^{(0)}, \qquad \frac{d\gamma_3^{(0)}}{dr} = \frac{r}{r^2 - a^2}\gamma_3^{(0)} \tag{4.4}$$

will be satisfied by the particular solutions

$$y_2^{(0)} = -\frac{\alpha}{c}r(r^2 - a^2)^{1/2}, \qquad \gamma_3^{(0)} = (r^2 - a^2)^{1/2} \tag{4.5}$$

To find the quantities $y_2^{(1)}$ and $\gamma_3^{(1)}$ let us substitute the relations (4.2), (4.3), (4.5) into the equation (3.9) and, making a transformation of variables according to the formulas

$$y_1^{(1)} = (r^2 - a^2)^{1/2}Y, \qquad \gamma_3^{(1)} = (r^2 - a^2)^{1/2}\Gamma$$

obtain the system

$$\frac{dY}{dr} = -\frac{\alpha}{c}\Gamma + \frac{1}{r^2 - a^2}\left[\frac{2\alpha^2a^4}{c(r^2 - a^2)} + \frac{\alpha^2r^2}{c} + \frac{2\alpha^2a^2}{c} + c\right], \qquad \frac{d\Gamma}{dr} = \frac{\alpha r}{r^2 - \alpha^2}\left(1 - \frac{2a^2}{r^2 - a^2}\right) \tag{4.6}$$

Integrating the last of equations (4.6) we will have

$$\Gamma = \frac{\alpha}{2}\ln(r^2 - \alpha^2) + \frac{a^2\alpha}{r^2 - a^2} + \text{const}$$

Substituting this expression into the first equation in (4.6), we obtain after integration

$$Y = -\frac{\alpha^2}{2c}r\ln(r^2 - a^2) + S\ln\frac{r - a}{r + a} + \varphi_1(r), \qquad S = \frac{1}{2a}\left(\frac{2\alpha^2a^2}{c} + c\right)$$

Here and in the following formulas $\varphi_i(r)$ $(i = 1, 2, \ldots)$ designates a certain algebraic function of r. Transferring back to the old variables, we will have

$$y_2^{(1)} = -\frac{\alpha^2}{2c} r (r^2 - a^2)^{1/2} \ln (r^2 - a^2) + S (r^2 - a^2)^{1/2} \ln \frac{r-a}{r+a} + \varphi_2 (r)$$
$$\gamma_3^{(1)} = \frac{\alpha}{2} (r^2 - a^2)^{1/2} \ln (r^2 - a^2) + \varphi_3 (r) \tag{4.7}$$

As pointed out above, if the initial system (3.1) has the fourth algebraic integral, then the system (3.9) must have the algebraic integral (3.12) which is reduced, by means of the relations (4.5) and (4.7), to the form

$$\left[S \frac{\partial F_0}{\partial y_2^{(0)}} \right] \ln \frac{r-a}{r+a} + \left[-\frac{\alpha^2}{2c} r \frac{\partial F_0}{\partial y_2^{(0)}} + \frac{\alpha}{2} \frac{\partial F_0}{\partial \gamma_3^{(0)}} \right] \ln (r^2 - a^2) = \varphi_4 (r) \tag{4.8}$$

Here, due to the relation (4.5) and the assumptions made about the function $F_0(y_2^{(0)}, \gamma_3^{(0)}, r)$, the expressions in brackets will be algebraic functions of r. The function $\ln(r^2 - a^2)$ cannot be represented [3] as an algebraic function of r and $\ln[(r - a)/(r + a)]$; hence, the left-hand side of the expression (4.8) can be an algebraic function of r if the expressions in brackets become equal to zero. From here follows the condition

$$\frac{\partial F_0}{\partial y_2^{(0)}} = \frac{\partial F_0}{\partial \gamma_3^{(0)}} = 0 \qquad (S \neq 0) \tag{4.9}$$

which must be satisfied for all r, except $r = \pm a$. But, by the property of the expression $F_0(y_2^{(0)}, \gamma_3^{(0)}, r)$ the condition (4.9) is not satisfied, which proves that in the case under consideration the existence of a general fourth algebraic integral for the system (3.1) or, which is the same, for the system (1.1) is impossible.

As shown in [6], in this case the problem does not have a general solution, single-valued on the entire plane t. Hence, the necessary and sufficient condition for the existence of the fourth independent general algebraic integral of the system (1.1) for $A = B$ is the condition $x_0' = y_0' = 0$.

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